On the exact solutions of fractional differentional equations using improved Riccati equations method

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Abstract— The main objective of this paper is to use the fractional complex transformation to convert the nonlinear partial fractional differential equations to the nonlinear ordinary differential equations. We showed that the fractional complex transformation is valid only in the case that fractional differential equations had general "wave" solutions. We used the improved Riccati equations method and obtained the exact solutions for nonlinear partial fractional differential equations. As an application we got the exact solutions for the space-time fractional generalized Zakharov equations and space-time fractional generalized Hirota–Satsuma coupled KdV equations. Also we gave a new solutions for the improved Riccati equations. This method is efficient and powerful in solving wide classes of nonlinear evolution fractional order equations. These explicit exact solutions contained solitary wave solutions, periodic wave solutions and the combined formal solitary wave solutions. The method can also be applied to solve more nonlinear partial differential equations. Finally, By comparison our new solutions by the other results we found that our solutions are new and not be found before.

Index Terms—Improved Riccati equations method, Generalized Zakharov equations, Generalized Hirota-Satsuma Coupled KdV Equations, fraction complex transform.

1 Introduction

Fractional differential equations are viewed as alternative models to nonlinear differential equations. Varieties of them play important roles and tools not only in mathematics but also in physics, dynamical systems, control systems and engineering to create the mathematical modeling of many physical phenomena. Furthermore, they employed in social science such as food supplement, climate and economics. Fractional differential equations concerning the Riemann-Liouville fractional operators or Caputo derivative have been recommended by many authors (see [1-5]). Transform is a significant technique to solve mathematical problems. Many useful transforms for solving various problems were appeared in open literature such as wave transformation, Laplace transform, the Fourier transform, the Bücklund transformation, the integral transform, the local fractional integral transforms, Mellin transform, and the fractional complex transform, which was first proposed by He and Li [6-12], among which the fractional complex transform [6–12] is the simplest approach, it is to convert the fractional differential equations into ordinary differential equations, making the solution procedure extremely simple. Similar to wave transformation which was to introduce in the form $\xi = qt + px + ky + lz$, where p, q, k and l are constants, for nonlinear wave equations, e. g., the KdV equation, the fractional complex transform also admits a complex variable ξ, instead of the above equation, defined as [6, 7]

$$\xi = \frac{qt^{\alpha}}{\Gamma(1+\alpha)} + \frac{px^{\beta}}{\Gamma(1+\beta)} + \frac{ky^{\gamma}}{\Gamma(1+\gamma)} + \frac{lz^{\delta}}{\Gamma(1+\delta)},$$

where α, β, γ and δ are fractional orders. Such transformation is valid only for general “wave” solutions for fractional differential equations. However, not every fractional differential equation has a “wave” solution, hence its application is limited, and for this reason He and Elagan et al. [13] suggested a general transform which depend on the fractal index. Assume that $f : R \to R, x \mapsto f(x)$ denote a continous (but not necessarily

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differentiable) function, and let h denote a constant discretization span, Jumarie's defined the fractional derivative in the limit form

$$f^{\alpha}(x)=\lim_{h\downarrow 0}\frac{\Delta^{\alpha}\left[f(x)-f(0)\right]}{h^{\alpha}},0<\alpha<1, \quad (1)$$

where

$$\Delta^{\alpha}f(x)=\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(\alpha-k+1)}f\left[x+(\alpha-k)h\right] \quad (2)$$

This definition is close to the standard definition of the derivative (calculus for beginners) and as a direct result, the $\alpha-$th derivative of a constant, $0<\alpha<1$, is zero. An alternative, which is the strictly equivalent to Eq. (1) is the following expression as

$$f^{\alpha}(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{0}^{x}(x-\xi)^{-\alpha}\left[f(\xi)-f(0)\right]d\xi,0<\alpha<1 \quad (3)$$

and $f^{\alpha}(x)=\left(f^{(n)}(x)\right)^{(\alpha-n)},n\le\alpha\le n+1,n\ge 1.$ (4)

Some properties of the fractional modified Riemann-Liouville derivative were summarized in, four useful formulas of them are

$$D_x^{\alpha}x^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha},\gamma>0, \quad (5)$$

$$D_x^{\alpha}\left(u(x)v(x)\right)=v(x)D_x^{\alpha}u(x)+u(x)D_x^{\alpha}v(x), \quad (6)$$

$$D_x^{\alpha}\left[f(u(x))\right]=f_u^{'}(u)D_x^{\alpha}u(x), \quad (7)$$

$$D_x^{\alpha}\left[f(u(x))\right]=D_u^{\alpha}f(u)\left(u_x^{'}\right)^{\alpha}. \quad (8)$$

which are direct consequences of the equality $d^{\alpha}x(t)=\Gamma(1+\alpha)dx(t)$ which holds for non-differentiable functions. In the above formulas (6)-(8), $u(x)$ is non-differentiable function in (6) and (7) and differentiable in (8), $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (7) and non-differentiable in (8). In this article, based on the fraction complex transform technique, we will devise effective way for solving fractional partial differentional equations. It will be shown that the use of the complex transform allows us to obtain new exact solutions from the known seed solutions for the time and space fractional generalized Zakharov equatios and space-time fractional generalized Hirota–Satsuma coupled KdV equations.

2 The Fractional Complex Transform

Consider the following nonlinear partial fractional differential equation:

$$F\begin{pmatrix} u,D_t^{\alpha}u,D_x^{\beta}u,D_y^{\gamma}u,D_z^{\delta}u, \\ D_t^{\alpha}D_t^{\alpha}u,D_t^{\alpha}D_x^{\beta}u,D_x^{\beta}D_x^{\beta}u,D_x^{\beta}D_y^{\gamma}u,D_y^{\gamma}D_y^{\gamma}u \end{pmatrix}=0 \quad (9)$$

$0<\alpha,\beta,\gamma,\delta<1$

where u is an unknown function, and F is a polynomial of u and its partial fractional derivatives, in which the highest order derivatives and the nonlinear terms are involved. In the following, we give the main steps of the improved Riccati equations method.

Step 1 He and Li [11] proposed a fractional complex transform to convert fractional differential equations into ordinary differential equations (ODE), so all analytical methods devoted to the advanced calculus can be easily applied to the fractional calculus. The traveling wave variable

$$u(x,y,z,t)=u(\xi),$$

$$\xi=\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{Ly^{\beta}}{\Gamma(1+\beta)}+\frac{Mz^{\gamma}}{\Gamma(1+\gamma)}\frac{Nt^{\delta}}{\Gamma(1+\delta)} \quad (10)$$

where K,L,M and N are non zero arbitrary constants, permits us to reduce Eq. (9) to an ODE of $u=u(\xi)$ in the form

$$P\left(u,u',u'',u''',\ldots\right)=0 \quad (11)$$

Step 2 ([14]) Suppose the solution of Eq. (3) is expressed in the general form

$$u(\xi)=\sum_{i=0}^{n}a_i f^{i}(\xi)+\sum_{j=0}^{n}b_i f^{j-1}(\xi)g^{j}(\xi), \quad (12)$$

Where $a_0,a_i,b_j\ (i,j=1,2,\ldots,n)$, are all constants to be de-

ISSN 2229-5518

termined later. The new variables $f(\xi), g(\xi)$ satisfy the following improved Riccati equations:

$$f'(\xi)=-qf(\xi)g(\xi), g'(\xi)=-q\left(1-g^2(\xi)-rf(\xi)\right),$$
$$g^2(\xi)=1-2rf(\xi)+\left(r^2+\varepsilon\right)f^2(\xi) \quad (13)$$

where $\varepsilon=\pm1, q\neq0, r$ are arbitrary constants. Now we obtain the solutions of (13). For $\rho(\xi)=\frac{1}{f(\xi)}$, we obtain the simple differential equation

$\rho''(\xi)=q^2\left(\rho(\xi)-r\right).$ The solution is $\rho(\xi)=c_1e^{q\xi}+c_2e^{-q\xi}+r.$ Therefore,

$$f(\xi)=\frac{1}{c_1e^{q\xi}+c_2e^{-q\xi}+r}, \text{ and}$$

$$g(\xi)=\frac{c_1qe^{q\xi}-c_2qe^{-q\xi}}{c_1e^{q\xi}+c_2e^{-q\xi}+r} \quad (14)$$

Note that: Exp-function Method is a special case of this method when $f(\xi)=\exp(q_1\xi)$ and $g(\xi)=\exp(q_2\xi)$.

Step 3 The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (11). (n is usually a positive integer). If n is a fraction or a negative integer , we make the following transformation: (a) when $n=\frac{d}{c}$ is a fraction, we take $u(\xi)=v^{\frac{d}{c}}(\xi)$, then return to determine the balance constant n again; (b) when n is a negative integer, we let $u(\xi)=v^n(\xi)$ then return to determine the balance constant n again.

Step 4 Substituting Eq. (12) into Eq. (11), using Eq. (13), separately yields a set of algebraic equations for $f^i(\xi)g^j(\xi)(i=1,2,...,j=0,1)$. Setting the coefficients of $f^i(\xi)g^j(\xi)$ to zero derives a set of over determined algebraic equations for $a_0, a_i, b_j\ (i,j=1,2,...,n)$, then solving this system using the Maple Package.

Step 5 Substitute $a_0, a_i, b_j\ (i,j=1,2,...,n)$ which are obtained in **step 4**, to Eq. (10), Eq. (11), Eq. (12) and Eq. (14) respectively. Then we obtain many exact solutions of Eq. (11).

3 Applications

In this section, we present two examples to illustrate the applicability of our method to solve nonlinear fractional partial differential equations.

Example 1. We first consider the space-time fractional generalized Zakharov equations for the complex envelope $\psi(x,t)$ of the high-frequency wave and the real low-frequency field $\varphi(x,t)$ in the following form

$$iD_t^a\psi(x)+D_x^{2a}\psi(x)-2\sigma\left|\psi(x)\right|^2\psi(x)+2\psi(x)\varphi(x)=0,$$
$$D_t^{2a}\varphi(x)-D_x^{2a}\varphi(x)+D_x^{2a}\left|\psi(x)\right|^2=0,\ 0<a\leq1 \quad (15)$$

where the cubic term in first equation of Eq. (15) describes the nonlinear-self interaction in the high frequency subsystem, such a term corresponds to a self-focusing effect in plasma physics. The coefficient σ is a real constant that can be a positive or negative number. To demonstrate the effectiveness of our approach, we apply the method to construct the exact solutions for the above equation. We can see that the fractional complex transform [6, 7]

$$\psi(x,t)=\Psi(\xi), \varphi(x,t)=\Phi(\xi), \xi=\frac{Kx^\alpha}{\Gamma(1+\alpha)}+\frac{Lt^\alpha}{\Gamma(1+\alpha)} \quad (16)$$

Where K and L are constants, permits us to reduce Eq. (7) into the following ODE:

$$iL\Psi'+K^2\Psi''-2\sigma\left|\Psi\right|^2\Psi+2\Psi\Phi=0,$$
$$L^2\Phi''-K^2\Phi''+K^2\Psi''=0, \quad 0<a\leq1. \quad (17)$$

By taking the pane we transformation in the form

$$\Psi(\xi)=\mathrm{H}(\xi)e^{i\xi} \quad (18)$$

where $\mathrm{H}(\xi)$ is a real function, we obtain the following ordinary differential equations

$$K^2 H'' - L H - K^2 H - 2\sigma H^3 + 2H\Phi = 0,$$

$$L H' + 2K^2 H' = 0$$

$$L^2\Phi'' - K^2\Phi'' + 2K^2 HH'' + 2K^2\left(H'\right)^2 = 0, \quad 0 < a \le 1. \quad (19)$$

where H, Φ satisfy Eq. (12) respectively. Considering the homogeneous balance between the highest order derivative and the nonlinear term in Eq. (19), we deduce that

$$H(\xi) = a_0 + a_1 f(\xi) + a_2 g(\xi),$$

$$\Phi(\xi) = b_0 + b_1 f(\xi) + b_2 f^2(\xi) + b_3 g(\xi) + b_4 f(\xi) g^2(\xi), \quad (20)$$

where $a_i, b_j\ (i = 0,1,2; j = 0,1,\ldots,4)$ are all constants to be determined later, and $f(\xi), g(\xi)$ satisfy Eq. (12) respectively. Substituting (20) with (13) into (19), the left hand side of Eq. (19) is converted into a polynomial of $f^i(\xi) g^j(\xi)\ (i = 0,1,\ldots,5; j = 0,1)$, then setting each coefficients to zero, we get a set of over-determined algebraic system with respect to the unknown $a_i, b_j\ (i = 0,1,2; j = 0,1,\ldots,4), K, L$. Solving the system of over-determined algebraic equations using Maple Package, we obtain the following sets of solutions.

Case 1

$$\varepsilon = \pm 1, r = 0, a_0 = a_1 = b_1 = b_3 = b_4 = 0, L = -2K^2, b_2 = -\frac{q^2 K^2 \varepsilon}{1 - \sigma + 4K^2\sigma},$$

$$a_2 = \pm\frac{Kq\sqrt{\left(1-\sigma+4K^2\sigma\right)\left(1-4K^2\right)}}{1-\sigma+4K^2\sigma}, b_0 = \frac{K^2\left(-1+\sigma-4K^2\sigma+8\sigma q^2K^2-2\sigma q^2\right)}{2\left(1-\sigma+4K^2\sigma\right)}. \quad (21)$$

Where $\sigma, q \ne 0, K \ne 0$ are all arbitrary constants, so according to Eqs. (14), (18), (20), (21), we obtain solitary wave solutions of Eq. (15) as follows

$$\psi(x,t) = \left(\pm\frac{Kq\sqrt{\left(1-\sigma+4K^2\sigma\right)\left(1-4K^2\right)}}{\sqrt{1-\sigma+4K^2\sigma}}x\right.$$

$$\left.\frac{c_1 q e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)} - c_2 q e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}}{c_1 e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)} + c_2 e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}}\right) e^{i\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}$$

$$\pm\left(\frac{Kq\sqrt{\left(1-\sigma+4K^2\sigma\right)\left(1-4K^2\right)}}{\sqrt{1-\sigma+4K^2\sigma}}x\right.$$

$$\left.\frac{c_1 q e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)} - c_2 q e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}}{c_1 e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)} + c_2 e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}}\right) e^{i\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}$$

, and

$$\varphi(x,t) = \frac{K^2\left(-1+\sigma-4K^2\sigma+8\sigma q^2K^2-2\sigma q^2\right)}{2\left(1-\sigma+4K^2\sigma\right)} \mp$$

$$\frac{q^2K^2}{1-\sigma+4K^2\sigma}\left(\frac{1}{c_1 e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)} + c_2 e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}}\right)^2.$$

Case 2

$$\varepsilon = -1, r = \pm 1, a_0 = a_1 = b_3 = 0, L = -2K^2, b_0 = \frac{K^2\left(-2+2\sigma-8K^2\sigma+4\sigma q^2K^2-\sigma q^2\right)}{4\left(1-\sigma+4K^2\sigma\right)}$$

$$b_2 = \pm 2b_4, b_1 = -\frac{\pm q^2K^2 - 2b_4 + 2\sigma b_4 - 8K^2\sigma b_4}{2\left(1-\sigma+4K^2\sigma\right)}, a_2 = \pm\frac{\sqrt{\left(1-\sigma+4K^2\sigma\right)\left(-1+4K^2\right)}}{2\left(1-\sigma+4K^2\sigma\right)}Kq,$$

where $\sigma, q \ne 0, K \ne 0, b_4$ are all arbitrary constants, so according to Eqs. (14), (18), (20), (21), we obtain solitary wave solutions of Eq. (15) as follows

$$\psi(x,t)=\left(\pm\frac{\sqrt{\left(1-\sigma+4K^{2}\sigma\right)\left(-1+4K^{2}\right)}}{\sqrt{2\left(1-\sigma+4K^{2}\sigma\right)}}Kq\right)x\left(\frac{c_1qe^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}-c_2qe^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm1}\right)e^{i\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}$$

, and

$$\varphi(x,t)=\frac{K^{2}\left(-2+2\sigma-8K^{2}\sigma+4\sigma q^{2}K^{2}-\sigma q^{2}\right)}{4\left(1-\sigma+4K^{2}\sigma\right)}\pm\frac{q^{2}K^{2}-2b_4+2\sigma b_4-8K^{2}\sigma b_4}{2\left(1-\sigma+4K^{2}\sigma\right)}x\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm1}\right)\pm\left(\frac{b_4}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm1}\right)x$$
$$\left(\frac{2}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm1}+\left(\frac{c_1qe^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}-c_2qe^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm1}\right)^{2}\right).$$

Case 3

$$r=0,a_0=a_2=b_1=b_3=b_4=0,L=-2K^{2},b_2=-\frac{q^{2}K^{2}\varepsilon}{1-\sigma+4K^{2}\sigma},$$

$$a_1=\pm\frac{Kq\sqrt{\left(1-\sigma+4K^{2}\sigma\right)\left(4K^{2}-1\right)\varepsilon}}{1-\sigma+4K^{2}\sigma},b_0=-\frac{1}{2}K^{2}-\frac{1}{2}K^{2}q^{2},\qquad(21)$$

where $\sigma,q\neq0,K\neq0$ are all arbitrary constants, so according to Eqs. (14), (18), (20), (21), we obtain solitary wave solutions of Eq. (15) as follows

$$\psi(x,t)=\left(\pm\frac{Kq\sqrt{\left(1-\sigma+4K^{2}\sigma\right)\left(4K^{2}-1\right)\varepsilon}}{\sqrt{1-\sigma+4K^{2}\sigma}}\right)x\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}\pm\varepsilon}\right)e^{i\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)},$$

and

$$\varphi(x,t)=-\frac{1}{2}K^{2}\left(1-q^{2}\right)-\frac{q^{2}K^{2}\varepsilon}{1-\sigma+4K^{2}\sigma}x\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}}\right)^{2}.$$

Example 2. Let us apply our method to the space-time fractional generalized Hirota–Satsuma coupled KdV equations

$$D_t^{a}u-\frac{1}{2}D_x^{3a}u+3uD_x^{a}u-3D_x^{a}(vw)=0,$$
$$D_t^{a}v+D_x^{3a}v-3uD_x^{a}v=0,$$
$$D_t^{a}w+D_x^{3a}w-3uD_x^{a}w=0\ ,\qquad 0<a\leq1.\qquad(22)$$

Equations (22) can be used to describe the interaction of two long waves with different dispersion relations. To demonstrate the effectiveness of our approach, we apply the method to construct the exact solutions for the above equation. We can see that the fractional complex transform

$$u(x,t)=U(\xi),v(x,t)=V(\xi),w(x,t)=W(\xi),$$
$$\xi=\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}+\frac{Lt^{\alpha}}{\Gamma(1+\alpha)}\ ,$$

Where K and L are constants, permits us to reduce Eq. (14) into the following ODE:

$$LU'-\frac{1}{2}K^{3}U'''+3KUU'-3K(VW)'=0,$$
$$LV'+K^{3}V'''-3KUV'=0,$$

$$LW'+K^3W'''-3KUW'=0. \tag{23}$$

Considering the homogeneous balance between the highest order derivative and the nonlinear term in Eq. (19), we deduce that

$$U(\xi)=a_0+a_1f(\xi)+a_2f^2(\xi)+a_3g(\xi)+a_4f(\xi)g^2(\xi),$$

$$V(\xi)=b_0+b_1f(\xi)+b_2f^2(\xi)+b_3g(\xi)+b_4f(\xi)g^2(\xi),$$

$$W(\xi)=c_0+c_1f(\xi)+c_2f^2(\xi)+c_3g(\xi)+c_4f(\xi)g^2(\xi), \tag{24}$$

where $a_i,b_i,c_i\ (i=0,1,\ldots,4)$ are all constants to be determined later, and $f(\xi),g(\xi)$ satisfy Eq. (12) respectively. Substituting (24) with (13) into (23), the left hand side of Eq. (23) is converted into a polynomial of $f^i(\xi)g^j(\xi)$, then setting each coefficients to zero, we get a set of over-determined algebraic system with respect to the unknown $a_i,b_i,c_i\ (i=0,1,\ldots,4),K,L$.

Solving the system of over-determined algebraic equations using Maple Package, we obtain the following sets of solutions.

Case 1:

$$a_0=\frac{L+K^3q^2}{3K},a_1=-2K^2q^2r,a_2=2K^2q^2r^2+2K^2q^2\varepsilon,$$

$$a_3=a_4=b_2=b_3=b_4=c_2=c_3=c_4=0,$$

$$b_0=\frac{Kq^2\left(4Lr^2+K^3c_1q^2r+4c_0Lr^2+K^3q^2c_0r^2+4c_0L\varepsilon-2K^3q^2c_0\varepsilon\right)}{3c_1^2},$$

$$b_1=\frac{K\left(4Lrc_1+K^3q^2r^2+4L\varepsilon K^3-2K^3q^2\varepsilon\right)}{3c_1},c_0=c_0,c_1=c_1,$$

we obtain solitary wave solutions of Eq. (22) as follows

$$u(x,t)=\frac{L+K^3q^2}{3K}-2K^2q^2r\,X\left(\frac{1}{c_1e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+r}\right)$$

$$+\left(2K^2q^2r^2+2K^2q^2\varepsilon\right)X\left(\frac{1}{c_1e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+r}\right)^2,$$

$$v(x,t)=\frac{Kq^2\left(4Lr^2+K^3c_1q^2r+4c_0Lr^2+K^3q^2c_0r^2+4c_0L\varepsilon-2K^3q^2c_0\varepsilon\right)}{3c_1^2}+$$

$$\left(\frac{K\left(4Lrc_1+K^3q^2r^2+4L\varepsilon K^3-2K^3q^2\varepsilon\right)}{3c_1}\right)X\left(\frac{1}{c_1e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+r}\right),$$

and

$$w(x,t)=c_0+\frac{c_1}{c_1e^{q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^\alpha}{\Gamma(1+\alpha)}-\frac{2K^2t^\alpha}{\Gamma(1+\alpha)}\right)}+r}.$$

Case 2

$$a_0=\frac{1}{4}\frac{\left(r^2+2\varepsilon\right)K^3q^2}{r^2+\varepsilon},a_1=-2K^2q^2r,a_2=2K^2q^2r^2+2K^2q^2\varepsilon,$$

$$a_3=a_4=b_1=b_2=b_3=b_4=c_2=c_3=c_4=0,$$

ISSN 2229-5518

$$b_0=-\frac{rq^4K^4\varepsilon}{c_1\left(r^2+\varepsilon\right)},L=-\frac{1}{4}\frac{K^3q^2\left(r^2-2\varepsilon\right)}{r^2+\varepsilon},c_0=c_0,c_1=c_1,K=K,$$

we obtain solitary wave solutions of Eq.(22) as follows

$$u\left(x,t\right)=\frac{1}{4}\frac{\left(r^2+2\varepsilon\right)K^3q^2}{r^2+\varepsilon}-2K^2q^2r\,X\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r}\right)$$
$$+\left(2K^2q^2r^2+2K^2q^2\varepsilon\right)X\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r}\right)^2,$$

$$v\left(x,t\right)=-\frac{rq^4K^4\varepsilon}{c_1\left(r^2+\varepsilon\right)},$$

and

$$w\left(x,t\right)=c_0+\frac{c_1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r}.$$

Case 3

$$a_0=\frac{L+K^3q^2}{3K},a_1=-2K^2q^2r,a_2=2K^2q^2r^2+2K^2q^2\varepsilon,$$

$$a_3=a_4=b_2=b_3=b_4=c_2=c_3=c_4=0,$$

$$c_0=-\frac{\left(4Lr^2b_0+K^3b_0q^2r^2+4b_1Lr+K^3q^2b_1r+4b_0L\varepsilon-2K^3q^2b_0\varepsilon\right)Kq^2}{3b_1^2},$$

$$c_1=\frac{Kq^2\left(K^3q^2r^2-2K^3q^2\varepsilon+4Lr^2++4L\varepsilon\right)}{3b_1},b_0=b_0,b_1=b_1,L=L,K=K,$$

we obtain solitary wave solutions of Eq.(22) as follows

$$u\left(x,t\right)=\frac{L+K^3q^2}{3K}-2K^2q^2r\,X\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r}\right)$$
$$+\left(2K^2q^2r^2+2K^2q^2\varepsilon\right)X\left(\frac{1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r}\right)^2,$$

$$v\left(x,t\right)=b_0+\frac{b_1}{c_1e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_2e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^2t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r},$$

and

$$w\left(x,t\right)=-\frac{\left(4Lr^2b_0+K^3b_0q^2r^2+4b_1Lr+K^3q^2b_1r+4b_0L\varepsilon-2K^3q^2b_0\varepsilon\right)Kq^2}{3b_1^2}+$$

$$\frac{Kq^{2}\left(K^{3}q^{2}r^{2}-2K^{3}q^{2}\varepsilon+4Lr^{2}++4L\varepsilon\right)}{\left(c_{1}e^{q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+c_{2}e^{-q\left(\frac{Kx^{\alpha}}{\Gamma(1+\alpha)}-\frac{2K^{2}t^{\alpha}}{\Gamma(1+\alpha)}\right)}+r\right)} \quad (3b)$$

4. Conclusions

The improved Riccati equation method is applied successfully for solving the system of nonlinear fractional differentional equations. The performance of this method is reliable and effective and gives more new solutions. This method has more advantages: it is direct and concise. Thus, we deduce that the proposed method can be extended to solve many systems of nonlinear fractional partial diffrentional equations.

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